Integrable Mappings for Non–Commutative Objects

A. N. Leznov* and E. A. Yuzbashyan†

Abstract

The integrable mappings formalism is generalized on non–commutative case. Arising hierarchies of integrable systems are invariant with respect to this "quantum" discrete transformations without any assumption about commutative properties of unknown operators they include. Partially, in the scope of this construction are the equations for Heisenberg operators of quantum (integrable) systems.

1 Introduction

In papers [1] was shown that the theory of integrable systems (under the assumption of commutativety of all involved functions) can be reformulated in a form, where the key role plays the group of integrable mappings and its theory of representation. It arose the question, what will happen with this construction, if we will consider equations of motion for Heisenberg operators, or in other words, when unknown functions of integrable systems changed on non–commutative variables?

The goal of the present paper is to give the partial answer to this question.

Each quantum system with the same success can be described in many different (in form) representations. The most known and used are Schrödinger and Heisenberg pictures. The first deal with the wave functions (the state vectors in a Hilbert space), the second with the non–commutative Heisenberg operators and equations of motion under appropriate initial conditions (commutation relations at the fixed moment of time).

In this paper we will show how the group of integrable mappings conception must be changed to include the non–commutative variables case. The equations of evolution type (after some modifications connected with the order of a multipliers) remain invariant with respect to the corresponding quantum discrete transformations without any assumption about commutation rules for unknown

*Institute for High Energy Physics,142284 Protvino, Moscow Region, Russia
†Joint Institute for Nuclear Research, 141980 Dubna, Moscow Region, Russia

functions (operators). Partially they can be $s \times s$ matrix functions or some operators acting in the arbitrary representation space.The equations of motion for quantum Heisenberg operators are containing within this construction.

We use the discrete transformations method as the most adequate to solution of such kind problems [1]. We restrict ourselves by some concrete examples of integrable mappings for non–commutative objects, in usual and supersymmetrical two–dimensional spaces and by corresponding hierarchies of $(1+2)$–dimensional integrable systems. We now have no idea how to enumerate all possible integrable mappings for non–commutative objects. In this connection we can only add that in commutative case this problem yet is very far From its final solution.

2 Non–Commutative Darboux–Toda Substitution in Two–Dimensional space

2.1 Definitions

Let us denote u, v the pair of operators defined in some representation space and depending on x, y coordinates of two–dimension space. We assume, that partial derivatives up to some sufficient large order and inverse operators u^{-1}, v^{-1} are exist. Only associativety is assumed for multiplication, nowhere we will assume any commutation relations.

We will consider the following mapping:

$$\overleftarrow{u} = v^{-1} \quad \overleftarrow{v} = [vu - (v_x v^{-1})_y]v \equiv v[uv - (v^{-1}v_y)_x], \tag{1}$$

where $\overleftarrow{u}; \overleftarrow{v}$ denotes final, transformed operators. In the case, when u, v are some $s \times s$ matrices (1) was considered in [2]. In classical case, when u, v are usual commutative functions (1) is the well-known Darboux–Toda substitution.

Substitution (1) is invertible, i.e. the initial operators can be expressed in the terms of final ones. Denoting $\overrightarrow{u}; \overrightarrow{v}$ the result of inverse transformation we have:

$$\overrightarrow{v} = u^{-1} \quad \overrightarrow{u} = [uv - (u_y u^{-1})_x]u \equiv u[vu - (u^{-1}u_y)_x] \tag{2}$$

Operator $f(u, v)$ after application of the s–times direct transformation we will denote $\overset{\leftarrow s}{f} \equiv f(\overset{\leftarrow s}{u}, \overset{\leftarrow s}{v})$ and after s–times inverse transformation as $\overset{s \rightarrow}{f} \equiv f(\overset{s \rightarrow}{u}, \overset{s \rightarrow}{v})$, with the agreement $\overset{\leftarrow -m}{f} \equiv \overset{m \rightarrow}{f}$, $m \geq 0$.
If

$$\begin{aligned} u_t &= F_1(u, v, u_x, v_x, u_y, v_y, ...) \\ v_t &= F_2(u, v, u_x, v_x, u_y, v_y, ...) \end{aligned} \tag{3}$$

is given evolution type system then the condition of its invariance with respect to the transformation (1) (it means that in terms of $\overleftarrow{u}, \overleftarrow{v}$ equations (3) will be

exactly the same as they are in terms of u, v) can be derived by differentiation of (1) by some parameter and has the following form:

$$\begin{array}{ll}\overleftarrow{F}_1=\overleftarrow{u}_t= & -v^{-1}v_tv^{-1}=-v^{-1}F_2v^{-1} \\ \overleftarrow{F}_2=\overleftarrow{v}_t= & ([vu-(v_xv^{-1})_y]v)_t=[F_2u+vF_1-(F_{2x}v^{-1})_y+ \\ & +(v_xv^{-1}F_2v^{-1})_y]v+[vu-(v_xv^{-1})_y]F_2\end{array} \tag{4}$$

This is the functional symmetry equation for substitution (1). Unknown operators here are F_1, F_2. If some operators $F_1(u,v), F_2(u,v)$ are solution of (4) then the corresponding system (3) will be invariant with respect to (1). (4) is a linear system, i.e. if $F_1{}', F_2{}'$ and $F_1{}'', F_2{}''$ are solutions then $F_1 = aF_1{}' + bF_1{}'', F_2 = aF_2{}' + bF_2{}''$, where a, b are arbitrary numerical parameters, is also solution. Every symmetry equation possesses trivial solution $F_1 = au_x + bu_y, F_2 = av_x + bv_y$. Substitution is called integrable if its symmetry equation have at least one non–trivial solution.

2.2 Solution of the Symmetry Equation

The method we will use here to find solutions of (4) is analogues to the method we used in [4] in the case of commutative functions. But it is not exactly the same as many transformations can not be done because of non–commutativety of variables under consideration.

First of all let us take $F_2 = \alpha_0 v$, $F_1 = u\beta_0$. We obtain:

$$\begin{array}{l}\beta_0=-\overrightarrow{\alpha}_0 \\ \alpha_{0xy}=(\alpha_0-\overleftarrow{\alpha}_0)\overleftarrow{T}_0+T_0(\alpha_0-\overrightarrow{\alpha}_0)+\theta\alpha_{0y},-\alpha_{0y}\theta\end{array} \tag{5}$$

where $T_0 = vu$, $\theta = v_xv^{-1}$. This system possesses obvious partial solution $\alpha_0{}^{(0)} = -\beta_0{}^{(0)} = 1$, which gives the first term of hierarchy: $F_1 = -u, F_2 = v$. Two following equations for T_0 and θ, which are the direct corollary of (1), are important for father calculations:

$$T_{0x}=\theta T_0-T_0\overrightarrow{\theta} \qquad \theta_y=T_0-\overleftarrow{T}_0 \tag{6}$$

In fact, (6) is the substitution (1) rewritten in terms of T_0 and θ. Now let us take $\alpha_{0y} = \alpha_1 \overleftarrow{T}_0 + T_0\beta_1$. One can treat this expression as analog to the decomposition of some vector by basic vectors. From (5), expressing $\overleftarrow{T}_{0x}$ and T_{0x} with the help of (6) and equating to zero coefficients before $\overleftarrow{T}_0$ and after T_0 (which is some additional assumption), we have:

$$\begin{array}{l}\alpha_{1x}=\alpha_0-\overleftarrow{\alpha}_0+\theta\alpha_1-\alpha_1\overleftarrow{\theta} \\ \beta_{1x}=\alpha_0-\overrightarrow{\alpha}_0+\overrightarrow{\theta}\beta_1-\beta_1\theta\end{array} \tag{7}$$

The second relation obviously can be rewritten as:

$$\overleftarrow{\beta}_{1x} = -(\alpha_0 - \overleftarrow{\alpha}_0) + \theta\, \overleftarrow{\beta}_1 - \overleftarrow{\beta}_1 \overleftarrow{\theta}$$

From what it follows that system (7) possesses partial solution of the form $\overleftarrow{\beta}_1 = -\alpha_1$. After taking y–derivative of equation for α_1, this partial solution gives the following system:

$$\begin{array}{l} \beta_1 = -\overrightarrow{\alpha}_1 \\ \alpha_{1xy} = (\alpha_1 - \overleftarrow{\alpha}_1)\, \overset{\leftarrow 2}{T_0} + T_0(\alpha_1 - \overrightarrow{\alpha}_1) + \theta\alpha_{1y} - \alpha_{1y}\, \overleftarrow{\theta} \end{array} \tag{8}$$

This system is analogues to the (5). It also has partial solution $\alpha_1 = -\beta_1 = 1$, which leads to ${\alpha_0}^{(1)} = \int(\overleftarrow{T}_0 - T_0)dy$, and gives the next solution of the symmetry equation (4). Taking now $\alpha_{1y} = \alpha_2\, \overset{\leftarrow 2}{T_0} + T_0\beta_2$ we are able to continue by the same scheme. The system for α_2, β_2 has the same structure as previous systems. Its partial solution $\alpha_2 = -\beta_2$ allows to find the third term of hierarchy

$${\alpha_0}^{(2)} = \int dy \left[\left(\int dy \left(\overset{\leftarrow 2}{T_0} - T_0 \right) \right) \overleftarrow{T}_0 - T_0 \int dy \left(\overleftarrow{T}_0 - \overrightarrow{T}_0 \right) \right]$$

By induction it can be proved that in general case equations for α_n, β_n have the form:

$$\begin{array}{l} \beta_n = -\overrightarrow{\alpha}_n \\ \alpha_{nxy} = (\alpha_n - \overleftarrow{\alpha}_n)\, \overset{\leftarrow (n+1)}{T_0} + T_0(\alpha_n - \overrightarrow{\alpha}_n) + \theta\alpha_{ny} - \alpha_{ny}\, \overset{\leftarrow n}{\theta} \end{array} \tag{9}$$

with partial solution $\alpha_n = -\beta_n = 1$. After this expression for ${\alpha_0}^{(n)}$ can be reconstructed in the form of the sum of 2^n terms, which can be written in the following symbolical form:

$$\begin{array}{rcl} {\alpha_0}^{(n)} & = & (-1)^n \prod_{i=1}^{n} \left(1 - L_i exp\left[id_i + \sum_{i=k+1}^{n} d_k \right]\right) \times \\ & & \times \int dy (T_0 \int dy (\overrightarrow{T}_0 \int dy (... \int dy \overset{(n-1)\rightarrow}{T_0} ...))) \end{array} \tag{10}$$

where $exp\, d_p$ means shifts by the unity the argument of p-repeated integral.

$$\ldots \int dy\, \overset{\leftarrow p}{T_0} \rightarrow \ldots \int dy\, \overset{\leftarrow (p+1)}{T_0} \ldots$$

and symbol L_r–transposition of terms in the r-th brackets

$$(A_1(\ldots(A_r[\ldots])\ldots)) \rightarrow (A_1(\ldots([\ldots]A_r)\ldots))$$

with the following multiplication rules:

$$L_i exp[...]_1 L_j exp[...]_2 = L_i L_j exp\,[[...]_1 + [...]_2]$$

Comparing (10) with [4] we see that here for non-commutativety we are forced to introduce the new operators L_i which are discount the order of the multipliers.

2.3 Examples

n=0

$$v_t = v \qquad u_t = -u$$

n=1

$$v_t = v_x \qquad u_t = u_x$$

n=2

$$v_t = v_{xx} - 2\int (vu)_x dy \times v \quad u_t = -u_{xx} + 2u \int (vu)_x dy$$

This is the Davey–Stewartson system, described in [3]

n=3

$$v_t = v_{xxx} - 3\int (vu)_x dy \times v_x - 3\int (v_x u)_x dy \times v -$$

$$-3\int \left[vu \int (vu)_x dy - \int (vu)_x dy \times vu \right] dy \times v$$

$$u_t = -u_{xxx} - 3u_x \int (vu)_x dy - 3u \int (v_x u)_x dy -$$

$$-3u \int \left[vu \int dy (vu)_x - \left(\int dy (vu)_x \right) vu \right] dy$$

In commutative case this is the Veselov–Novikov system.

3 Non–Commutative Darboux–Toda Transformation in Two–Dimensional Super Space

3.1 Definitions

Here we will analyze the situation, when non-commutative operators under consideration in addition to usual space and time coordinates x, y, t are depend upon Grassman variables θ_+, θ_-. We will consider the following mapping:

$$\overleftarrow{u} = v^{-1} \quad \overleftarrow{v} = -[D_-(D_+ v \times v^{-1}) + vu]v \equiv v[D_+(v^{-1}D_- v) - uv], \tag{11}$$

where

$$D_+ = \frac{\partial}{\partial \theta_+} + \theta_+ \frac{\partial}{\partial x} \quad D_- = \frac{\partial}{\partial \theta_-} + \theta_- \frac{\partial}{\partial y} \quad D_+^2 = \frac{\partial}{\partial x} \quad D_-^2 = \frac{\partial}{\partial y}$$

Other notations are the same as in the previous section. Substitution (11) is invertible. Inverse transformation has the form:

$$\overrightarrow{v} = u^{-1} \quad \overrightarrow{u} = -[D_+(D_-u \times u^{-1}) + uv]u \equiv u[D_-(u^{-1}D_+u) - vu], \tag{12}$$

The symmetry equation for (11) is the following:

$$\begin{array}{rcl} \overleftarrow{F_1} &=& -v^{-1}F_2v^{-1} \\ \overleftarrow{F_2} &=& F_2[D_+(v^{-1}D_-v) - uv] + v[D_+(-v^{-1}F_2v^{-1}D_-v) + \\ && +D_+(v^{-1}D_-F_2) - F_1v - uF_2] \end{array} \tag{13}$$

3.2 Solution of the Symmetry Equation

Here we will get the hierarchy of solutions of the symmetry equation (13). For this we will use the same general method as in the previous section. But there is an interesting and in some sense important difference. As we will see bellow, partial solutions of (13) can be found only at even steps, when unknown operators are Bosonic–like variables, whereas at odd steps they are Fermionic–like.

After substitution in (13) $F_1 = u\beta_0, F_2 = \alpha_0 v$ we have:

$$\begin{array}{l} \beta_0 = -\overrightarrow{\alpha_0} \\ D_+D_-\alpha_0 = (\overleftarrow{\alpha}_0 - \alpha_0)\overleftarrow{T_0} + T_0(\alpha_0 - \overrightarrow{\alpha_0}) + \theta D_-\alpha_0 + D_-\alpha_0\theta, \end{array} \tag{14}$$

where $T_0 = vu, \theta = D_+v \times v^{-1}$. This system has partial solution $\alpha_0 = -\beta_0 = 1$, which correspond to: $F_1 = -u, F_2 = v$. Transformation (11) can be rewritten in terms of T_0, θ as:

$$D_+T_0 = \theta T_0 - T_0\overrightarrow{\theta} \qquad D_-\theta = -T_0 - \overleftarrow{T_0} \tag{15}$$

Taking now $D_-\alpha_0 = \alpha_1\overleftarrow{T_0} + T_0\beta_1$, for α_1, β_1 we have:

$$D_+\alpha_1 = \overleftarrow{\alpha}_0 - \alpha_0 + \theta\alpha_1 + \alpha_1\overleftarrow{\theta}$$

$$D_+\beta_1 = \alpha_0 - \overrightarrow{\alpha}_0 + \overrightarrow{\theta}\beta_1 + \beta_1\theta$$

For $\overleftarrow{\beta}_1 = \alpha_1$ the second relation directly follows from the first one. For this case, acting on the equation for α_1 with D_- operator, we have:

$$\begin{array}{l} \beta_1 = \overrightarrow{\alpha_1} \\ -D_+D_-\alpha_1 = (\alpha_1 + \overleftarrow{\alpha}_1)\overleftarrow{T_0}^2 - T_0(\alpha_1 + \overrightarrow{\alpha_1}) + D_-\alpha_1\overleftarrow{\theta} - \theta D_-\alpha_1 \end{array} \tag{16}$$

This is the typical system for odd steps. Comparing it with (8) we notice that the difference between those systems is that (16) have not numerical partial solutions (α_1, β_1 are Fermionic–like operators). However it is possible to continue

reduction using decomposition $D_-\alpha_1 = \alpha_2 \overset{\leftarrow 2}{T_0} + T_0\beta_2$. We have:

$$-D_+\alpha_2 = \overleftarrow{\alpha}_1 + \alpha_1 - \theta\alpha_2 + \alpha_2 \overset{\leftarrow 2}{\theta}$$

$$-D_+\beta_2 = -(\alpha_1 + \overrightarrow{\alpha_1}) - \overrightarrow{\theta}\,\beta_2 + \beta_2 \overleftarrow{\theta}$$

Taking $\overleftarrow{\beta}_2 = -\alpha_2$, after usual simple calculations we will find:

$$\begin{aligned} &\beta_2 = -\overrightarrow{\alpha_2} \\ &D_+D_-\alpha_2 = (\overleftarrow{\alpha}_2 - \alpha_2) \overset{\leftarrow 3}{T_0} + T_0(\alpha_2 - \overrightarrow{\alpha_2}) + D_-\alpha_1 \overset{\leftarrow 2}{\theta} + \theta D_-\alpha_2 \end{aligned} \tag{17}$$

The partial solution of this system is: $\alpha_2 = -\beta_2 = 1$; it correspond to the trivial system $F_1 = au_x + bv_y, F_2 = av_x + bv_y$. All systems received on even steps will be similar to (17). Partial solution of each next system of that kind gives non–trivial, nonlinear evolution type system invariant with respect to the transformation (11) (see $k = 2$ example). By induction easily can be proved that for arbitrary $n = 2k + 1$ we will have:

$$D_-\alpha_{n-1} = \alpha_n \overset{\leftarrow n}{T_0} + T_0 \overrightarrow{\alpha}_n$$

$$D_-\alpha_n = \alpha_{n+1} \overset{\leftarrow n+1}{T_0} - T_0 \overrightarrow{\alpha}_{n+1}$$

$$\begin{aligned} D_+D_-\alpha_{n-1} = (\overleftarrow{\alpha}_{n-1} - \alpha_{n-1}) \overset{\leftarrow n}{T_0} - T_0(\alpha_{n-1} - \overrightarrow{\alpha}_{n-1}) + \\ + D_-\alpha_{n-1} \overset{\leftarrow n-1}{\theta} + \theta D_-\alpha_{n-1} \end{aligned} \tag{18}$$

$$-D_+D_-\alpha_n = (\overleftarrow{\alpha}_n + \alpha_n) \overset{\leftarrow n+1}{T_0} - T_0(\alpha_n + \overrightarrow{\alpha}_n) + D_-\alpha_n \overset{\leftarrow n}{\theta} - \theta D_-\alpha_n$$

After this using $\alpha_{2k} = 1$ partial solution of the system (18) it is possible to construct the k–th term of hierarchy. One can prove using induction that the final result can be represented as:

$$\begin{aligned} \alpha_0{}^{(k)} \quad = \quad & (-1)^k \prod_{i=1}^{2k} \left(1 - (-1)^i L_i exp\left[id_i + \sum_{i=k+1}^{2k} d_k\right]\right) \times \\ & \times D_-^{-1}(T_0 D_-^{-1}(\overrightarrow{T_0}\, D_-^{-1}(...D_- - 1 \overset{(n-1)\rightarrow}{T_0} ...))) \end{aligned} \tag{19}$$

The meaning of notations here is the same as in formula (10).

3.3 Examples

k=0

$$v_t = v \qquad u_t = -u$$

k=1

$$v_t = v_x \qquad u_t = u_x$$

k=2

$$v_t = v_{xx} - 2D_-^{-1}(vu)_x D_+ v - 2D_-^{-1}(vD_+u) \times v +$$
$$+2D_-^{-1}\left[vuD_-^{-1}(vu)_x + D_-^{-1}(vu)_x \times vu\right]$$

$$u_t = -u_{xx} + D_+ u D_-^{-1}(vu)_x - 2uD_-^{-1}(D_+vu)_x -$$
$$-2uD_-^{-1}\left[vuD_-^{-1}(vu)_x + D_-^{-1}(vu)_x \times vu\right]$$

4 Conclusion

The main concrete result of the paper is the explicit form of quantum integrable systems (10), (19) in the mentioned above sense.

It is interesting that the scheme of our calculations is similar to the computer program algorithm–there are many identical operations with possibility to interrupt them at arbitrary step. Obviously, in this scheme is coded the structure of the group of integrable mappings, more exactly one of the possible connections between the integrable system by itself and its symmetry equation. If it will be possible to translate it on the group–theoretical language, then we will be near to understand the integrable substitutions role and near to the classification theorem for them.

It is well known that quantum integrable systems are closely connected with so–called "quantum" algebras [5]. Moreover this object of mathematics in essential part was discovered and developed under the investigations of the integrable systems in quantum region.

So it arise more wide, deep and interesting problem–to find the connection between the approach of this paper and sufficiently developed formalism of quantum algebras. The equations for Heisenberg operators, as it was mentioned in the introduction, are only one of the possible representations of the quantum picture. We hope that further investigations will find some bridge connecting quantum integrable mappings of the present paper with quantum algebras of the traditional approach. But now we are not ready and able to go so far and hope to return to this problem in the future publications.

5 Acknowledgments

The authors wish to thank the Russian Foundation for Fundamental Researches for partial support trough the Grant 95–01–00249

References

[1] D.B. Fairlie, A.N. Leznov// Phys. Lett. A 199 (1995) 360-364

A.N. Leznov// Physica D 87 (1995) 48-51

[2] Leznov A.N., Yuzbashyan E.A.//*Lett. in Math. Phys. 35:345-349, 1995*

[3] Leznov A.N., Yuzbashyan E.A.–Preprint MPI–96-37, Bonn 1996.

[4] Leznov A.N., Yuzbashyan E.A.–Preprint IHEP–95-28, Protvino 1995

[5] V.G. Drinfeld "Quantum Groups" In: Proceedings of the Intern. Congress of Mathematicians Vol. 1, pp. 798-820. New York: Berkley 1986 (AMS 1987)

L.D. Fadeev, N.Yu. Reshetikhin, L.A. Takhtajan Algebra&Analiz 1(89)178 English translation in: Leningrad Math. J. 1(90)193